Einstein Metrics and Smooth Structures on Non-Simply Connected 4-Manifolds

Ioana Şuvaina

May 1, 2008

Abstract

We analyze the existence or non-existence of Einstein metrics on 4-manifolds with non-trivial fundamental group and the relation with the differential structure considered. We conclude that for admissible pairs (n, m) in a large region of the integer lattice, the manifold $n\mathbb{CP}^2\#m\overline{\mathbb{CP}^2}$ admits infinitely many non-equivalent free actions of finite cyclic groups and there are no Einstein metrics which are group invariant.

The main tools are Seiberg-Witten Theory, cyclic branched coverings of complex surfaces and symplectic surgeries.

1 Introduction

The classical obstructions to the existence of an Einstein metric on a four-manifold are topological. If (M, g) is a smooth, compact, oriented four-manifold, endowed with an Einstein metric g, then its Euler characteristic, $\chi(M)$, must be non-negative and it satisfies the Hitchin-Thorpe Inequality:

$$(2\chi \pm 3\tau)(M) \geq 0 \tag{1}$$

where $\tau(M)$ is the signature of M.

Using Seiberg-Witten equations, LeBrun [LeB96, LeB01] found obstructions to the existence of Einstein metrics on a large class of manifolds for which the topological obstructions are satisfied. These obstructions provided the first means of exhibiting the strong dependence of the existence of Einstein metrics on the differential structure of the underlying four-manifold. Examples of pairs of homeomorphic, but not diffeomorphic, simply connected manifolds such that one manifold admits an Einstein metric while

the other does not, were first found by Kotschick [Kot98]. Later on, the obstructions were improved by LeBrun (see [LeB01]) and new examples were constructed. His main theorem is the following:

Theorem 1.1. [LeB01] *Let X be a compact oriented 4-manifold with a non-trivial Seiberg-Witten invariant and with $(2\chi + 3\tau)(X) > 0$. Then*

$$M = X\# k\overline{\mathbb{CP}^2}\# l(S^1 \times S^3)$$

does not admit Einstein metrics if $k + 4l \geq \frac{1}{3}(2\chi + 3\tau)(X)$.

In [IsLe02], LeBrun and Ishida raised the question of the number of smooth structures of a topological 4-manifold for which no compatible Einstein metric exists. They show that there are infinitely many such structures.

All known results are for simply-connected manifolds. In this paper we will analyze the case of non-simply connected manifolds.

Our first theorem is about manifolds with arbitrary finite cyclic fundamental group:

Theorem 1.2. *For any finite cyclic group $\mathbb{Z}/d\mathbb{Z}, d > 1$, there exist infinitely many pairs of compact oriented smooth 4-manifolds $(Z_i, \{M_{i,j}\}_{j\in\mathbb{N}})_{i\in\mathbb{N}}$, all having fundamental group $\mathbb{Z}/d\mathbb{Z}$ and satisfying:*

1. *For i fixed, any two manifolds in $\{Z_i, M_{i,j}\}$ are homeomorphic, but no two are diffeomorphic to each other;*

2. *Z_i admits an Einstein metric, while no $M_{i,j}$ admits an Einstein metric.*

Moreover their universal covers $\widetilde{Z}_i$ and $\widetilde{M_{i,j}}$ satisfy:

3. *$\widetilde{M_{i,j}}$ is diffeomorphic to $n\mathbb{CP}^2\# m\overline{\mathbb{CP}}^2$, where $n = b_2^+(\widetilde{Z}_i)$ and $m = b_2^-(\widetilde{Z}_i)$;*

4. *$\widetilde{Z}_i$ and $\widetilde{M_{i,j}}$ are not diffeomorphic, but become diffeomorphic after taking the connected sum with one copy of $\mathbb{CP}^2$.*

For suitable n, m on $n\mathbb{CP}^2\# m\overline{\mathbb{CP}}^2$, $n > 1$, it was showed [IsLe02, BrKo05] that there are infinitely many distinct **exotic** structures for which no Einstein metric exists. But there is not much known about the canonical smooth structure. In some cases, for manifolds yielded by constructions similar to the ones in the proof of Theorem 1.2, we can show that there are no Einstein metrics which are invariant under the action of $\mathbb{Z}/d\mathbb{Z}$. To be more precise, two of the examples with the smallest topology would be the following:

Proposition 1.3. *On* $15\mathbb{CP}^2\#77\overline{\mathbb{CP}^2}$*, there exists an involution* σ*, acting freely on the manifold, such that* $15\mathbb{CP}^2\#77\overline{\mathbb{CP}^2}$ *does not admit an Einstein metric invariant under the involution* σ*.*

As we increase the degree of the action, the numerical invariants rise fast:

Proposition 1.4. *On* $23\mathbb{CP}^2\#116\overline{\mathbb{CP}^2}$*, there exists a free* $\mathbb{Z}/3\mathbb{Z}-$ *action, such that* $23\mathbb{CP}^2\#116\overline{\mathbb{CP}^2}$ *does not admit a* $\mathbb{Z}/3\mathbb{Z}-$*invariant Einstein metric.*

For arbitrary n, m we can formulate a more general statement:

Theorem 1.5. *For any small* $\epsilon > 0$ *there exists an* $N(\epsilon) > 0$ *such that for any integer* $d \geq 2$ *and any lattice point* (n, m)*, satisfying:*

1. *Hitchin Thorpe Inequality:* $n > 0$
2. *admissibility condition:* $d/n, d/m$
3. $n < (6 - \epsilon)m - N(\epsilon)$

there exist infinitely many, non-equivalent, free $\mathbb{Z}/d\mathbb{Z}-$*actions on* $X = (2m-1)\mathbb{CP}^2\#(10m-n-1)\overline{\mathbb{CP}^2}$ *(i.e* $(2\chi + 3\tau)(X) = n, \frac{\chi+\tau}{4}(X) = m)$*. Moreover, there is no Einstein metric on* X *invariant under any of the* $\mathbb{Z}/d\mathbb{Z}-$*actions.*

One important ingredient in the proof of the above result is the geography of almost completely decomposable symplectic manifolds, due to Braungardt and Kotschick [BrKo05].

The $\mathbb{Z}/d\mathbb{Z}-$actions are such that the quotient manifolds are homeomorphic, but not diffeomorphic. The differential structures are distinguished by their Seiberg-Witten invariants. As a consequence of the non-triviality of these invariants any invariant constant scalar curvature metric on X must have non-positive constant.

Remark The result in Theorem 1.5 holds for finite cyclic groups, and also for groups acting freely on the 3-dimensional sphere or for direct sums of the above groups. All one has to do is to substitute any of the above groups instead of $\mathbb{Z}/d\mathbb{Z}$ in the proof of the theorem. $\square$

In general, the existence or non-existence of Einstein metrics on an arbitrary manifold is hard to prove. On the remaining part of this paper we will emphasize the non-existence of Einstein metrics on non-simply connected manifolds.

Theorem 1.6. *Let G be any finitely presented group. There exists an infinite sequence of non-homeomorphic, non-spin $4-$manifolds $M_i, i \in \mathbb{N}$, with fundamental group G such that each M_i supports infinitely many distinct smooth structures $M_{i,j}, j \in \mathbb{N}$, which do not admit an Einstein metric. All $M_{i,j}$ satisfy Hitchin-Thorpe Inequality.*

The manifolds constructed in Theorems 1.2,1.6 are necessarily non-spin. However, using obstructions (Theorem D [IsLe03]) induced by a non-trivial Bauer-Furuta invariant, one can also find examples of spin manifolds.

Theorem 1.7. *For any finite group G, there is an infinite family of spin $4-$manifolds M_i with fundamental group $\pi_1(M_i) = G$ such that each space has infinitely many, distinct smooth structures, which do not admit an Einstein metric. All these manifolds satisfy the Hitchin-Thorpe Inequality.*

In some cases, for a small fundamental group $\mathbb{Z}/d\mathbb{Z}$, some of the manifolds constructed in the theorem support a differential structure which admits a Kähler-Einstein metric, but most of the manifolds M_i don't support a complex structure.

On simply connected spin manifolds it was proved that there are no Einstein metrics for some exotic smooth structures. We can prove non-existence of $\mathbb{Z}/d\mathbb{Z}-$invariant Einstein metrics for the canonical smooth structure on certain connected sums of $K3$'s and $S^2 \times S^2$'s.

Theorem 1.8. *There exists an $n_0 > 0$ such that for any $d > n_0$ the manifolds:*

1. $M_{1,n} = d(n+5)(K3)\#(d(n+7)-1)(S^2 \times S^2)$

2. $M_{2,n} = d(2n+5)(K3)\#(d(2n+6)-1)(S^2 \times S^2)$

$n \in \mathbb{N}^*$, *admit infinitely many non-equivalent free $\mathbb{Z}/d\mathbb{Z}$ actions, such that there is no Einstein metric on $M_{1,n}, M_{2,n}$ invariant under any of the $\mathbb{Z}/d\mathbb{Z}-$actions.*

The coefficients are chosen such that the topological invariants are divisible by d.

The paper is organized as follows: in Section 2 we review some background results on the topology of $4-$manifolds, in the third section we give a construction of manifolds of general which admit free $\mathbb{Z}/d\mathbb{Z}$ actions and in Section 4 we give the proofs of the above theorems.

Acknowledgments. I would like to thank my advisor Professor Claude LeBrun for suggesting this problem to me.

2 Background results on the topology of 4-manifolds

2.1 Homeomorphism criteria

A remarkable result of Freedman [Fre82] in conjunction with results of Donaldson tells us that smooth compact, simply connected, oriented 4-manifolds are classified by their numerical invariants: Euler characteristic χ, signature τ and Stiefel-Whitney class w_2. Maybe not as well-known are the results involving the classification of *non-simply connected* 4-manifolds ([HaKr93], Theorem C):

Theorem 2.1 (Hambleton, Kreck)**.** *Let M be a smooth, closed, oriented, 4-manifold with finite cyclic fundamental group. Then M is classified up to homeomorphism by the fundamental group, the intersection form on $H_2(M,\mathbb{Z})/Tors$, and the w_2-type. Moreover, any isometry of the intersection form can be realized by a homeomorphism.*

In contrast with simply connected manifolds, there are three w_2-types that can be exhibited: (I) $w_2(\widetilde{M}) \neq 0$, (II) $w_2(M) = 0$, and (III) $w_2(\widetilde{M}) = 0$, but $w_2(M) \neq 0$.

Using Donaldson's and Minkowski-Hasse's classification of the intersection form we can reformulate this theorem on an easier form:

Equivalently: A smooth, closed, oriented 4-manifold with finite cyclic fundamental group and indefinite intersection form is classified up to homeomorphism by the fundamental group, the numbers $b_2^{\pm}$, the parity of the intersection form and the w_2-type.

We would like to draw the reader attention on the fact that in the presence of 2-torsion one must be careful to determine both the parity of the intersection form and the w_2-type. There are known examples of non-spin manifolds with even intersection form, see for example D.Acosta and T.Lawson's paper.

However, on a 4-manifold with finite fundamental group, knowing the invariants $b_2^{\pm}$ is equivalent to knowing any other two numerical invariants, for example the Euler characteristic χ and signature τ.

2.2 Almost complete decomposability

Freedman's results tell us that any non-spin simply connected 4-manifold is homeomorphic to $a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$, for appropriate positive integers a, b. Using non-triviality of the Seiberg-Witten invariants, infinitely many differential

structures can be exhibited on many manifolds which admit a smooth, symplectic structure. In the case of $b_2^+ > 1$, such manifolds are never diffeomorphic to $a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$ as the invariants of the later manifold vanish. One way of measuring how different the differential structures are, is by taking connected sum with a copy of $\mathbb{CP}^2$. In this case, the Seiberg-Witten invariants of both manifolds vanish by Taubes' theorem. Hence it is natural to introduce the following definitions, due to Mandelbaum and Moishezon:

Definition 1. *We say a smooth non-spin* 4*-manifold* M *is* completely decomposable *if it is diffeomorphic to a connected sum of* $\mathbb{CP}^2$*'s and* $\overline{\mathbb{CP}^2}$*'s.*

Definition 2. *We say* M *is* almost completely decomposable (ACD) *if* $M\#\mathbb{CP}^2$ *is completely decomposable.*

Almost completely decomposable manifolds are quite common. It has been conjectured by Mandelbaum [Ma80] that simply connected analytic surfaces are almost completely decomposable. In support of this conjecture he proved that any complex surface which is diffeomorphic to complete intersection of hypersurfaces in some $\mathbb{CP}^N$ is almost completely decomposable. And also:

Proposition 2.2. [Ma80] *Let* $X \subset \mathbb{CP}^N$ *be a compact complex surface and suppose* $M \longrightarrow X$ *is an* r*-fold cyclic branched cover of* X *whose branch locus is homeomorphic to* $X \cap H_r$*, for some hypersurface* H_r *of degree* r *of* $\mathbb{CP}^N$*. Then if* X *is almost completely decomposable so is* M*.*

For a definition of a r-fold cyclic cover see the next section.

Using iterated cyclic-covers of $\mathbb{CP}^2$ and symplectic surgeries, Braungardt and Kotschick, [BrKo05], were able to show the abundance of the ACD symplectic manifolds for fixed topological invariants:

Theorem 2.3. [BrKo05] *For every* $\epsilon > 0$*, there is a constant* $c(\epsilon) > 0$ *such that every lattice point* (x, y) *in the first quadrant satisfying*

$$y \leq (9 - \epsilon)x - c(\epsilon)$$

is realized by the Chern invariants (χ_h, c_1^2) *of infinitely many pairwise non-diffeomorphic simply connected minimal symplectic manifolds, all of which are almost completely decomposable.*

3 A construction of surfaces of general type with finite cyclic fundamental group

One technique which yields a large family of examples of complex manifolds is the construction of finite cyclic covers.

For more details on the constructions presented in this chapter we refer the reader to [BPV84] for an algebraic geometric point of view, or to [GoSt99] for a more topological description.

Definition 3. *A (non-singular)* d-fold branched cover *consists of a triplet* (X, Y, π), *denoted by* $\pi : X \to Y$, *where* X, Y *are connected compact smooth complex manifold and* π *is a finite, generically* $d : 1$, *surjective, proper, holomorphic map.*

The critical set, $R \subset X$, is called the *ramification divisor* of π and its image $D = \pi(R)$ is called the *branch locus.* For any point $y \in Y \setminus D$ there is a connected neighborhood V_y with the property that $\pi^{-1}(V_y)$ consists of d disjoint subsets of X, each of which is mapped isomorphically onto V_y by π.

3.1 Cyclic covers

One special class of covers are the *cyclic covers.* They are constructed as follows:

Construction 1. : *Let* Y *be a connected complex manifold and* D *an effective divisor on* Y. *Let* $\mathcal{O}(D)$ *be the associated line bundle and* $s_D \in \Gamma(Y, \mathcal{O}_Y(D))$ *the section vanishing exactly along* D. *Suppose we have a line bundle* $\mathcal{L}$ *on* Y *such that* $\mathcal{O}_Y(D) = \mathcal{L}^{\otimes d}$. *We denote by* L *the total space of* $\mathcal{L}$ *and we let* $p : L \to Y$ *be the bundle projection. If* $z \in \Gamma(L, p^*\mathcal{L})$ *is the tautological section, then the zero divisor of* $p^*s_D - z^d$ *defines an analytical subspace* X *in* L. *If* D *is a smooth divisor then* X *is smooth connected manifold and* $\pi = p|_X$ *exhibits* X *as a* d*-fold ramified cover of* Y *with branch locus* D. *We call* (X, Y, π) *the* d*-cyclic cover of* Y *branch along* D, *determined by* $\mathcal{L}$.

Given D and Y, X is uniquely determined by a choice of $\mathcal{L}$. Hence X is uniquely defined if $Pic(Y)$ has no torsion.

A *cyclic* branched cover is a d-fold cover such that $\pi_{|X\setminus R} : X \setminus R \to Y \setminus D$ is a (regular) cyclic covering. Hence it is determined by an epimorphism $\pi_1(Y \setminus D) \to \mathbb{Z}_d$, and $Y = X/\mathbb{Z}_d$. Moreover, a cyclic d-cover is a Galois

covering, meaning that the function field embedding $\mathbb{C}(Y) \subset \mathbb{C}(X)$ induced by π is a Galois extension.

The following lemmas give us the main relations between the two manifolds:

Lemma 3.1. [BPV84] *Let $\pi : X \to Y$ be a d-cyclic cover of Y branched along a smooth divisor D and determined by $\mathcal{L}$, where $\mathcal{L}^{\otimes d} = \mathcal{O}_Y(D)$. Let R be the reduced divisor $\pi^{-1}(D)$ on X. Then:*

(i) $\mathcal{O}_X(R) = \pi^*\mathcal{L}$;

(ii) $\pi^*[D] = d[R]$, *in particular d is the branching order along R*;

(iii) $\mathcal{K}_X = \pi^*(\mathcal{K}_Y \otimes \mathcal{L}^{d-1})$.

As an immediate consequence, we are able to compute the relations between the topological invariants of X and Y in the case of complex surfaces:

Lemma 3.2. *Let X, Y complex surfaces and $\pi : X \to Y$ be as in Lemma 3.1. Then:*

(i) $c_2(X) = dc_2(Y) - (d-1)\chi(D)$;

(ii) $c_1^2(X) = d(c_1(Y) - (d-1)c_1(\mathcal{L}))^2$

In a more general set-up, we can define a d-cyclic branch cover $\pi : X \to Y$ branched along a divisor with simple normal crossing singularities and Y smooth manifold. In this case, X will be a normal complex surface ([BPV84] I.17.) with singularities over the singular points of D. Let $U \subset Y$ be a neighborhood of a singular point of D and (x, y) local coordinates such that D is defined by the equation $xy = 0$. Then $\pi^{-1}(0)$ is an isolated singularity of X and an open neighborhood of $\pi^{-1}(0) \in \pi^{-1}(U) \subset X$ is modelled in local coordinates by $z^d = xy \subset \mathbb{C}^3$. This type of singularity is known as an A_d-singularity. There are two techniques to associate a smooth manifold to X. One is given by resolving the singularities, the second is smoothing.

Given a normal surface X there is always a bi-meromorphic map $\pi : X' \to X$, with X' smooth. Moreover, if we require that X' is a minimal surface, then X' is uniquely determined by X (see for instance [BPV84] III Theorems 6.1, 6.2). $\pi : X' \to X$ is called the *minimal resolution* of singularities of X.

Definition 4. *A smoothing of a normal surface X is a proper flat map $f : \mathcal{X} \to \Delta$, smooth over $\Delta^* = \Delta \setminus 0$ where: $\mathcal{X}$ is a three dimensional complex manifold, Δ is a small open disk in $\mathbb{C}$ centered at 0, $f^{-1}(0)$ is isomorphic to X and $f^{-1}(t), t \neq 0$ smooth.*

If $t, t' \in \Delta^*$ then $f^{-1}(t)$ is diffeomorphic to $f^{-1}(t')$. In the case of cyclic branced coveringgs, a stronger result is true:

Proposition 3.3. *If $\pi : X \to Y$ is a $d-$cyclic cover branched along a divisor D with simple normal crossing singularities, such that the linear system $\mathbb{P}\big(H^0(Y, \mathcal{O}(D))\big)$ is base point free. Then, there is a smoothing $\varpi : \mathcal{X} \to \Delta$ of X and the generic fiber $X_t = \varpi^{-1}(t)$ is diffeomorphic to the minimal resolution X' of X.*

Proof. We will give the complete proof for double covers and then argue using the properties of local deformations of A_d-singularities [HKK86] that the same is true in general.

First, we remark that X has a finite number of singular points, corresponding to the singular points of D. Resolving these singularities is a local process. The singularity, modeled by $(z^2 = xy) \subset \mathbb{C}^3$, is a quotient singularity. It is isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action is just multiplication by (-1). The isomorphism is given by the map:

$$\mathbb{C}^2/\mathbb{Z}_2 \to U, \quad \widehat{(u,v)} \mapsto (u^2, v^2, uv) = (x, y, z).$$

Hence, it is enough to resolve the singularities of type $\mathbb{C}^2/\mathbb{Z}_2$. A resolution of this singularity is given by blowing-up the origin of $\mathbb{C}^2$ extending the $\mathbb{Z}_2$ action trivially on the exceptional divisor and then considering the new quotient space. The total space of the blow-up of $\mathbb{C}^2$ is, in fact, the line bundle $\mathcal{O}_{\mathbb{CP}^1}(-1)$, and factoring by the above $\mathbb{Z}_2$ action corresponds to squaring (tensor product) the line bundle. The resulting manifold, after taking the quotient, is $\mathcal{O}_{\mathbb{CP}^1}(-2)$. So, we resolved the singularities of X by introducing exceptional divisors of self-intersection (-2).

Next, we explicitly construct a smoothing of X. The idea is smoothing the branch locus in a family of smooth curves and constructing the corresponding double-covers. As the linear system $\mathbb{P}\big(H^0(Y, \mathcal{O}(D))\big)$ is base point free, so there exists a holomorphic path of sections of $\mathcal{O}(D)$ and a parametrization of this path given by $\varphi : \Delta \to \Gamma(Y, \mathcal{O}(D))$ such that $\varphi(0) = \varphi_D$ and $d\varphi_{|_{0,SingD}} \neq 0$. The last condition just says that the parametrization is "nice", i.e. the curve $\varphi(t) = 0, t \neq 0$, doesn't contain any of the singularities of D. Maybe after restricting Δ, we can also assume that $\varphi(t) = \varphi_t, t \neq 0$ corresponds to a smooth divisor.

Let $\mathcal{X} \subset \mathcal{L} \times \Delta$ given locally by the equation $z^2 - \varphi(t)(x, y) = 0$, where $\mathcal{L} \to Y$ is a line bundle such that $\mathcal{L}^2 = \mathcal{O}(D)$ and z is a local coordinate on $\mathcal{L}$. Then $\varpi : \mathcal{X} \to \Delta$ is a smoothing of $X = \varpi^{-1}(0)$. First, let's notice that

$\mathcal{X}$ is a smooth manifold as:

$$d(z^2-\varphi(t)(x,y)) = 2zdz-(\frac{d}{dt}\varphi(t)(x,y))dt-\Big(\frac{d\varphi_t}{dx}(x,y)dx+\frac{d\varphi_t}{dy}(x,y)dy\Big) \neq 0$$

This is never zero as for $t \neq 0$ the section φ_t is smooth, hence the last parenthesis is non-zero and for $t = 0$ we have $\frac{d\varphi}{dt} \mid_{0,SingD} \neq 0$ and $(\frac{d\varphi}{dx} + \frac{d\varphi}{dy}) \mid_{0,D\setminus SingD} \neq 0$.

As an immediate consequence of Theorem 9.11 in [Har77] the morphism ϖ is flat.

The fact that the two constructions yield diffeomorphic manifolds is a local statement about the differential structures of the new manifolds in a neighborhood of the singularities. So, our proof is in local coordinates. Because the morphism ϖ is a submersion away from the central fiber it is enough to show that one of the fibers is diffeomorphic to X'.

In local coordinates the singularities are given by the equation

$$(z^2 - xy = 0) \subset \mathbb{C}^3.$$

Because the linear system associated to $\mathcal{O}(D)$ is base point free, then the zero locus of a generic section is smooth. We can consider preferred local coordinates such that the smoothing is given by $(z^2 - xy = 1) \subset \mathbb{C}^3$. If we change the local coordinates $(x, y, z) \to (u, v, z)$, such that $x = iu - v,\ y = iu+v$ then the smoothing is written in the canonical form $(z^2+u^2+v^2 = 1)$. Let $\xi = Re(u, v, z),\ \eta = Im(u, v, z),\ \xi, \eta \in \mathbb{R}^3$ the real, imaginary part, of (u, v, z). Then:

$X_1 = \{\ (u, v, z) \in \mathbb{C}^3 \mid z^2 + u^2 + v^2 = 1\ \}$
$\quad = \{\ (\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid\ \parallel \xi \parallel^2 - \parallel \eta \parallel^2 = 1, < \xi, \eta >= 0\ \}$.

The map $f : X_1 \to T^*S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ defined by $f(\xi, \eta) = (\frac{\xi}{\|\xi\|}, \parallel \xi \parallel \eta)$ is a orientation preserving diffeomorphism. It is a well known fact that T^*S^2 is diffeomorphic to $\mathcal{O}_{\mathbb{CP}^1}(-2)$, where $\mathbb{CP}^1$ is identified to the sphere S^2.

To prove the result for a $d-$cyclic cover, the extra ingredient needed is that the local smoothing of the singularities is diffeomorphic to its resolution. For singularities of type $z^d = xy$, or type A_{d-1}, the required diffeomorphism is proved by Harer, Kas and Kirby, [HKK86], by using topological Kirby calculus. □

A similar statement is true for double covers branched along a divisor D with *simple singularities*, i.e. the singularities are double or triple points with two, three different tangents or simple triple points with one tangent. The double cover branched along such a divisor has $A - D - E$ singularities,

respectively. We call these singularities *rational double points*. It can be proved [HKK86] that for these singularities, and only for these singularities, the resolution and smoothing manifolds are diffeomorphic.

Next we want to study the fundamental group of our manifolds. We need to introduce a new definition:

Definition 5. *A smooth divisor D is said to be flexible if there exists a divisor $D' \equiv D$ such that $D \bigcap D' \neq \emptyset$ and D' intersects D transversally in codimension* 2.

We remark that if D is a flexible divisor then it must be connected.

Then, reformulating Catanese's Proposition 1.8 from [Cat84] in our easy situation, we have:

Proposition 3.4. [Cat84] *Let $\pi : X \to Y$ a d-cyclic cover branched along a smooth flexible divisor D, then if Y is simply connected so is X.*

Let's notice that if the branch locus is not flexible then the covering manifold might not be simply connected: consider for example the double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along four vertical $\mathbb{CP}^1$.

3.2 Bi-cyclic covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$

In this section we introduce a new construction which we are going to call the simple bi-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then, we study the analytical and topological properties of this class of manifolds. They are inspired by a construction due to Catanese. In his papers [Cat84], [Cat92], etc, he extensively studies two successive double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$. For our purposes we need to consider cyclic covers of arbitrary degrees. We have the following construction:

Construction 2. *Let C, D be two smooth transversal curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ such that the line bundles $\mathcal{O}(C), \mathcal{O}(D)$ are d, p-tensor powers of some line bundles on $\mathbb{CP}^1 \times \mathbb{CP}^1$. We construct a new manifold N by taking: first the d-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along C, which we denote by X, and then the p-cyclic cover of this manifold branched along the proper transform of D. We have the following diagram:*

$$N \to X \to \mathbb{CP}^1 \times \mathbb{CP}^1$$

It can easily be checked that if C, D are smooth and intersect transversally, then both X and N are smooth. We call the manifold N a (simple) bi-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ of type (d, p) branched along (C, D).

Let $\pi_1, \pi_2 : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ be the projections on the first, second factor, respectively. The line bundles on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are of the form $\pi_1^*(\mathcal{O}_{\mathbb{CP}^1}(a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{CP}^1}(b))$, which we denote by $\mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(a,b)$ or simply $\mathcal{O}(a,b)$. With this notations, $\mathcal{O}(C) = \mathcal{O}(da, db)$ and $\mathcal{O}(D) = \mathcal{O}(pm, pn)$ where a, b, m, n are non-negative integers. Let $\varphi_C \in \Gamma(\mathcal{O}(C))$ and $\varphi_D \in \Gamma(\mathcal{O}(D))$ such that $C = \{\varphi_C = 0\}$ and $D = \{\varphi_D = 0\}$. Let E be the total space of $\mathcal{E} = \mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(a,b) \oplus \mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(m,n)$ and $\pi : E \to \mathbb{CP}^1 \times \mathbb{CP}^1$. Then the manifold N can also be seen as a smooth compact submanifold of E. If $(z, w) \in \Gamma(E, \pi^*\mathcal{E})$ is the tautological section, then N is defined by the following equations $\{z^d - \pi^*\varphi_C = 0,\ w^p - \pi^*\varphi_D = 0\}$. As an immediate consequence we obtain that Construction 2 is commutative, i.e. we obtain the manifold N also as the type (p, d) bi-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along (D, C).

Let π be the projection from N to $\mathbb{CP}^1 \times \mathbb{CP}^1$ induced by the fibration projection. Using Lemma 3.2, we can easily compute the topological invariants of N.

Lemma 3.5. *Let $\pi : N \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be a bi-cyclic cover as above. Then:*

(i) $K_N = \pi^*\Big(\mathcal{O}\Big(\ (d-1)a + (p-1)m - 2\ ,\ (d-1)b + (p-1)n - 2\Big)\Big)$;

(ii) *If* $(d-1)a + (p-1)m \geq 3$ *and* $(d-1)b + (p-1)n \geq 3$ *then* K_N *is an ample line bundle;*

(iii) $c_1^2(N) = 2pd\Big((d-1)a + (p-1)m - 2\Big)\Big((d-1)b + (p-1)n - 2\Big)$;

(iv) $c_2(N) = pd[4 - 2(d-1)(a + b - dab) - 2(p-1)(m + n - pmn) + (p-1)(d-1)(an + bm)]$.

Proof. The proof of (i,iii,iv) is an immediate consequence of Lemmas 3.1, 3.2. The extra ingredients that need to be computed are the Euler characteristics of the branch loci C, D, $\chi(C) = 2d((a+b) - dab)$, $\chi(D) = 2p((m+n) - pmn)$, and $D' \subset X$ the proper transform of D. D' is a $d-$cover of D branched on $\mathrm{card}(C \cap D) = dp(an + bm)$ points. Then: $\chi(D') = d\chi(D) - (d-1)(C \cdot D) = pd(2(m + n - pmn) - (d-1)(an + bm))$.

K_N is the pull-back of an ample line bundle through a finite map hence, see [Har77], K_N is ample. $\square$

Next we analyze the topological properties of our manifolds.

Lemma 3.6. *If $\pi : N \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be a bi-cyclic cover as above and a, b, m, n are strictly positive integers, then N is simply connected.*

Proof. If a, b, m, n are strictly positive integers, then the divisors C, D are flexible divisors and applying Proposition 3.4 twice gives us that the manifold N is simply connected. $\square$

As N is a Kähler manifold it has non-trivial Seiberg-Witten invariants. Hence it does not decompose as connected sums of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}^2}$'s. At most we can hope that it will decompose after taking the connected sum with a copy of $\mathbb{CP}^2$. We have a more general statement:

Theorem 3.7. *Iterated cyclic covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along smooth, flexible curves, such that any two curves intersect transversally, are almost completely decomposable.*

Proof. We prove the theorem by double induction on the number of covers and the degree of the last cover. First we show that the d-cover $\pi_2 : X \to \mathbb{CP}^1 \times \mathbb{CP}^1$ branched along C is almost completely decomposable. Let $\varphi_{(a,b)} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^P$, where $P = (a+1)(b+1) - 1$, be the Segre-type embedding, and $[p_{ij}]$ be homogeneous coordinates on $\mathbb{CP}^P$ corresponding to (a, b)-bi-homogeneous monomials. Then if φ_C is the bi-degree (da, db) polynomial whose zero locus is C, then $\varphi_C = \varphi^*_{(a,b)}(f(p_{ij}))$, where f is a degree d polynomial. Hence, by Proposition 2.2, X is almost completely decomposable.

We need one more ingredient. The following lemma is proved in [MaMo80]:

Lemma 3.8. ([MaMo80], 3.4) *Suppose W is a compact complex 3-manifold and V, X_1, X_2 are closed simply-connected complex submanifolds with normal crossing in W. Let $S = X_1 \bigcap X_2$ and $C = V \bigcap S$. Suppose as divisors V is linearly equivalent to $X_1 + X_2$ and that $C \neq \emptyset$. Set $n = \operatorname{card} C$ and g be the genus of S. Then we have the diffeomorphism:*

$$V \# \mathbb{CP}^2 \cong X_1 \# X_2 \# \, 2g\, \mathbb{CP}^2 \# (2g + n - 1)\overline{\mathbb{CP}^2}.$$

For the last step of induction, let $\pi_2 : X \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be an iterated cyclic cover and let $\pi_1 : N \to X$ be the p-cyclic cover branched along $D' = \pi_2^{-1}(D)$. Then $\mathcal{O}(D') = \pi_2^*(\mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(pm, pn))$ and let $\mathcal{L} = \pi_2^*(\mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(m, n))$. Assuming the notations from Construction 1, there exists a tautological section $z \in \Gamma(L, p^*(\mathcal{L}))$ such that $N \subset L$ is the zero locus of $z^p - p^*(\varphi_{D'}) \in \Gamma(L, p^*(\mathcal{L}^{\otimes p}))$. We can compactify L to $W = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X)$ by adding a divisor $W_\infty = W \setminus L$. Then $p' : W \to X$ is a $\mathbb{CP}^1$-bundle and we can try to extend the section z to a section in $\Gamma(W, p'^*(\mathcal{L}))$. As $p'^*(\mathcal{L})$ restricts on the fiber to the trivial line bundle and z has a zero

at the origin of $L \to X$ then this section will have a pole of multiplicity one along W_∞. To adjust to this problem we consider the section $z' \in \Gamma(W, p'^*(\mathcal{L}) \otimes \mathcal{O}(W_\infty))$, $z' = z \cdot \varphi_{W_\infty}$. Let $\varphi_{W_\infty} = \varphi_\infty$. Then z' has no zero along W_∞ and rescales the values of z outside W_∞. Hence, our manifold N is the zero locus ($z'^p - p'^*(\varphi_D) \cdot \varphi_\infty^p = 0$).

Let D_1, D_2 be two smooth curves on $\mathbb{CP}^1 \times \mathbb{CP}^1$, such that $\mathcal{O}(D_1) = \mathcal{O}(m,n), \mathcal{O}(D_2) = \mathcal{O}((p-1)m, (p-1)n)$ and which are transversal to each other and to the branch loci which where considered in the construction of X and such that none of the intersection points of D_1, D_2 lie on the previous branch loci.

Then $p'^{-1}(D_i) = D_i'$, $i = 1,2$ are smooth curves on X transversal to each other and $D_1' + D_2'$ is linearly equivalent to D'. Let

$$X_1 = (z' - p'^*(\varphi_{D_1}) \cdot \varphi_\infty = 0)$$

$$X_2 = (z'^{p-1} - p'^*(\varphi_{D_2}) \cdot \varphi_\infty^{p-1} = 0)$$

We remark that X_1 is a cover of X of degree one, hence it is diffeomorphic to X, while X_2 is a $(p-1)-$cyclic cover of X. Then N, X_1, X_2 verify the requirements in the lemma so:

$$N \# \mathbb{CP}^2 \cong X \# X_2 \# r\mathbb{CP}^2 \# s\overline{\mathbb{CP}^2}, \text{ for suitable } r, s.$$

If $R = (\varphi_{D_1}^{p-1} - \varphi_{D_2} = 0) \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. Then $X_1 \bigcap X_2 = S = (\pi_2 \circ p')^{-1}(R)$ is an iterated cover of R. After taking the first branch cover we obtain a curve which is the $d-$cover of R branched at $R \cdot C = dam + dbn$ points, so its Euler characteristic is $d\,\chi(R) - (d-1)d(am+bn) \leq 0$. Hence genus of S is strictly greater than zero, i.e. $r \geq 1$.

If $p = 2$ then both $X_{1,2}$ are diffeomorphic to X hence almost completely decomposable, which implies N almost completely decomposable by Lemma 3.8. The induction on degree of the cover finishes the proof. □

3.3 Free actions of the finite cyclic group

On the remaining part of this section we give a recipe for constructing complex surfaces of general type with fundamental group isomorphic to $\mathbb{Z}/d\mathbb{Z}$. They are quotients of a free $\mathbb{Z}/d\mathbb{Z}$ action on bi-cyclic covers $\pi : N \to \mathbb{CP}^1 \times \mathbb{CP}^1$ of type (p, d). We construct the action of $\mathbb{Z}/d\mathbb{Z}$ explicitly.

First we consider actions on an arbitrary line bundle $\mathcal{O}(a, b)$.
Let $x = [x_0 : x_1], y = [y_0 : y_1]$ be homogeneous coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

We want to consider the action of $\mathbb{Z}_d =< e^{\frac{2\pi i}{d}} >$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ generated by:

$$e^{\frac{2\pi i}{d}} * ([x_0 : x_1], [y_0 : y_1]) = ([e^{\frac{2\pi i}{d}} x_0 : x_1], [e^{\frac{2\pi i}{d}} y_0 : y_1]).$$

This action has four fixed points:

$$([0:1],[0:1]),\ ([0:1],[1:0]), ([1:0],[0:1]),\ ([1:0],[1:0]).$$

Let $U_0 = \{[1 : x_1] | x_1 \in \mathbb{C}\}$ and $U_1 = \{[x_0 : 1] | x_0 \in \mathbb{C}\}$ be charts of $\mathbb{CP}^1$. Then the charts $U_0 \times U_0, U_0 \times U_1, U_1 \times U_0, U_1 \times U_1$ form an atlas of $\mathbb{CP}^1 \times \mathbb{CP}^1$. The line bundle $\mathcal{O}(a,b)$ restricted to each of these charts admits a trivialization. Let $z_{00}, z_{01}, z_{10}, z_{11}$ be the corresponding coordinates on each trivialization.

On the chart $U_1 \times U_1 \times \mathbb{C}$, we let the $\mathbb{Z}_d =< e^{\frac{2\pi i}{d}} >$ action be generated by:

$$e^{\frac{2\pi i}{d}} * ([x_0 : x_1], [y_0 : y_1], z_{11}) \to ([e^{\frac{2\pi i}{d}} x_0 : x_1], [e^{\frac{2\pi i}{d}} y_0 : y_1], e^{\frac{2\pi i}{d}} z_{11})$$

Using the change of coordinates, the above action is generated in the other charts by:

$$\begin{aligned}
&\text{on } U_0 \times U_1: \ (x_1, y_0, z_{01}) \to (e^{\frac{2\pi i(d-1)}{d}} x_1,\ e^{\frac{2\pi i}{d}} y_0,\ e^{\frac{2\pi i(1+a)}{d}} z_{01});\\
&\text{on } U_1 \times U_0: \ (x_0, y_1, z_{10}) \to (e^{\frac{2\pi i}{d}} x_0,\ e^{\frac{2\pi i(d-1)}{d}} y_1,\ e^{\frac{2\pi i(1+b)}{d}} z_{10});\\
&\text{on } U_0 \times U_0: \ (x_1, y_1, z_{00}) \to (e^{\frac{2\pi i(d-1)}{d}} x_1,\ e^{\frac{2\pi i(d-1)}{d}} y_1,\ e^{\frac{2\pi i(1+a+b)}{d}} z_{00}).
\end{aligned}$$

Hence, we have the following lemma:

Lemma 3.9. *If each of the integers $a+1,\ b+1,\ a+b+1$ is relatively prime to d, then the above action of $\mathbb{Z}/d\mathbb{Z}$ on $\mathcal{O}(a,b)$ is semi-free with four fixed points on the $0-$section.*

We can also consider a weighted action of $\mathbb{Z}_d$ which is defined on $U_1 \times U_1 \times \mathbb{C}$ as follows: $e^{\frac{2\pi i}{d}} * (x_0, y_0, z_{11}) \to (e^{\frac{2\pi i}{d}} x_0, e^{2\frac{2\pi i}{d}} y_0, e^{\frac{2\pi i}{d}} z_{11})$. This action extends on the other coordinate charts as:

$$\begin{aligned}
&\text{on } U_0 \times U_1: \ (x_1, y_0, z_{01}) \to (e^{\frac{2\pi i(d-1)}{d}} x_1,\ e^{2\frac{2\pi i}{d}} y_0,\ e^{\frac{2\pi i(1+a)}{d}} z_{01});\\
&\text{on } U_1 \times U_0: \ (x_0, y_1, z_{10}) \to (e^{\frac{2\pi i}{d}} x_0,\ e^{\frac{2\pi i(d-2)}{d}} y_1,\ e^{\frac{2\pi i(1+2b)}{d}} z_{10});\\
&\text{on } U_0 \times U_0: \ (x_1, y_1, z_{00}) \to (e^{\frac{2\pi i(d-1)}{d}} x_1,\ e^{\frac{2\pi i(d-2)}{d}} y_1,\ e^{\frac{2\pi i(1+a+2b)}{d}} z_{00}).
\end{aligned}$$

We have a similar lemma:

Lemma 3.10. *If each of the integers $a+1,\ 2b+1,\ a+2b+1$ is relatively prime to d, then the above action of $\mathbb{Z}/d\mathbb{Z}$ on $\mathcal{O}(a,b)$ outside the four points on the $0-$section is free.*

Requiring that a smooth flexible divisor $D \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ be invariant under the first $\mathbb{Z}/d\mathbb{Z}$ action is equivalent to $\varphi_D = 0$ is $\mathbb{Z}/d\mathbb{Z}$ invariant. But we need a stronger condition, we need to consider divisors D such that φ_D is a $\mathbb{Z}/d\mathbb{Z}$ invariant polynomial. Then φ_D is a

bi-homogeneous polynomial of bi-degrees divisible by d. So, there are strictly positive integers (a,b) such that $\mathcal{O}(D) = \mathcal{O}(da,db)$ and

$$\varphi_D = \sum_{\substack{i=\overline{0,a}\\ j=\overline{0,b}}} a_{ij} X_0^{di} X_1^{d(a-i)} Y_0^{dj} Y_1^{d(b-j)} + \sum_{i=\overline{0,da}} \sum_{\substack{j\\ i\leq dj\leq db+i}} b_{ij} X_0^{i} X_1^{da-i} Y_0^{dj-i} Y_1^{d(b-j)+i}$$

for some complex coefficients a_{ij}, b_{ij}. The linear system of $\mathbb{Z}/d\mathbb{Z}-$invariant sections of $\mathcal{O}(da,db)$ is base point free, hence by Bertini's Theorem the generic section is smooth, and we can choose D such that none of the four points are on D. Hence $\mathbb{Z}/d\mathbb{Z}$ acts freely on D.

If we consider the weighted action then the class of $\mathbb{Z}/d\mathbb{Z}-$invariant divisors might be larger then the one described above (if d even). But this is not important from our point of interest.

Let $N' \to \mathbb{CP}^1 \times \mathbb{CP}^1$ a $d-$cyclic cover branched along D with $\mathcal{O}(D) = \mathcal{O}(da,db)$ and such that φ_D is $\mathbb{Z}/d\mathbb{Z}-$invariant. Then N' is a submanifold in $\mathcal{O}(a,b)$ and the $\mathbb{Z}/d\mathbb{Z}$-action on $\mathcal{O}(a,b)$ restricts to N'. Moreover, if D does not contain any of the fixed four points and the conditions in the Lemma 3.9 are satisfied, then $\mathbb{Z}/d\mathbb{Z}$ acts freely on N'.

Unfortunately, for what we need, the above construction is not good enough and we will need to consider bi-cyclic covers.

Now, we are ready to construct our examples:

Proposition 3.11. *Let C, D smooth, flexible, transversal divisors on $\mathbb{CP}^1 \times \mathbb{CP}^1$ such that both φ_D and φ_C are invariant under the $\mathbb{Z}/d\mathbb{Z}$ action and $\mathbb{Z}/d\mathbb{Z}$ acts freely on D. Let a,b,m,n be positive integers such that $\mathcal{O}(D) = \mathcal{O}(da,db), \mathcal{O}(C) = \mathcal{O}(pdm,pdn)$, and d is relatively prime to each of the integers $a+1, b+1, a+b+1$. Then, the bi-cyclic cover $\pi : N \to \mathbb{CP}^1 \times \mathbb{CP}^1$ of type (d,p) branched along (D,C) admits a free $\mathbb{Z}/d\mathbb{Z}$ action. The quotient $M = N/(\mathbb{Z}/d\mathbb{Z})$ has the following properties:*

(i) *M is a smooth complex surface, with fundamental group $\pi_1(M) = \mathbb{Z}/d\mathbb{Z}$;*

(ii) K_M *is an ample line bundle if* $(d-1)a+d(p-1)m > 2$ *and* $(d-1)b+d(p-1)n > 2$;

(iii) $c_1^2(M) = 2p\Big((d-1)a + d(p-1)m - 2\Big)\Big((d-1)b + d(p-1)n - 2\Big)$;

(iv) $c_2(M) = p[4 - 2(d-1)(a+b-dab) - 2d(p-1)(m+n-pdmn) + 2(d-1)d(p-1)(an+bm)]$.

Proof. First, we need to define the $\mathbb{Z}/d\mathbb{Z}$ action on N. The action that we want to consider is the trivial extension of the action on $\mathcal{O}(a,b)$ to an action on $\mathcal{O}(a,b) \oplus \mathcal{O}(dm,dn)$. The conditions from the theorem imply that the action restricts to N as a free holomorphic action. Its quotient is a smooth complex surface, with fundamental group $\pi_1(M) = \mathbb{Z}/d\mathbb{Z}$. The numerical invariants of N are described by Lemma 3.5. The invariants of M are related to those of N by the following relations: $c_2(M) = \frac{1}{d}c_2(N)$, $c_1^2(M) = \frac{1}{d}c_1^2(N)$. The computations in (iii,iv) are immediate.

By Lemma 3.5, K_N is ample which implies [Har77] that so is K_M. □

If X is a complex surface and $\mathcal{O}_X$ the structure sheaf of X then we denote by $\chi_h(X) = \chi(X, \mathcal{O}_X)$ its *holomorphic Euler characteristic.* Todd-Hirzebruch formula tells us that this is the same as the *Todd genus* of our manifold X. It can be easily computed [BPV84] in terms of the Chern invariants as $\chi_h(X) = \frac{c_1^2(X)+c_2(X)}{12} = \frac{(\chi+\tau)}{4}(X)$. On manifolds with finite fundamental group any two numerical invariants completely determine the others. Because the holomorphic Euler characteristic is constant under the blow-up process we prefer to use this invariant instead of the Euler characteristic. The conditions that d and $a+1, b+1, a+b+1$ are relatively prime integers imply that $\chi_h(M)$ is an integer number, as expected.

The techniques developed in this section allow us to construct a family of manifolds with special topological properties:

Proposition 3.12. *Given any integer* $d \geq 2$, *there are infinitely many complex surfaces of general type,* $\{Z_i\}_{i\in\mathbb{N}}$, *with ample canonical line bundle, such that their fundamental groups* $\pi_1(Z_i) = \mathbb{Z}/d\mathbb{Z}$, *they have* $w_2 -$ *type* I, *odd intersection form, and* $c_1^2(Z_i) < 5\chi_h(Z_i)$. *Moreover, their universal covers are almost completely decomposable manifolds.*

Proof. We need to consider two cases, determined by the parity of d.

For d odd we have the following construction: Let $M(d; a, b, m, n)$ the manifold constructed in Proposition 3.11 as a $\mathbb{Z}_d$-quotient of a bi-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ of type $(d, 2)$, branched along (D, C) with $\mathcal{O}(D) = \mathcal{O}(da, db)$,

$\mathcal{O}(C) = (2dm, 2dn)$. Let $Z_i = M(d; d, d, i, i)$. As $d+1, 2d+1$ are relatively prime to d, the conditions in Proposition 3.11 are satisfied. An easy computation of the numerical invariants of Z_i yields:

- $c_1^2(Z_i) = 4(d(d-1) + di - 2)^2$;
- $\chi_h(Z_i) = \frac{1}{3}d^2(d-1)(2d-1) + d(d-1)(di-1) + d^2i^2 - 2di + 2$.

We can compute the signature in terms of these invariants as $\tau(Z_i) = (-8\chi_h + c_1^2)(Z_i)$. Rohlin's Theorem states that on a spin manifold we have the following relation: $\tau \equiv 0 \bmod 16$. For i odd, $c_1^2(Z_i) \neq 0 \bmod 8$, hence $\tau(Z_i) \neq 0 \bmod 8$. If d odd, then the first homotopy has no $2-$torsion hence Z_i non-spin implies odd intersection form ([Gom95]5.7.6). Its universal cover $\widetilde{Z}_i$ has signature $\tau(\widetilde{Z}_i) = d\tau(Z_i)$, so for d, i odd numbers $\tau(\widetilde{Z}_i) \neq 0 \bmod 16$. Hence Z_i is of w_2-type (I) and odd intersection form.

As i increases, $\frac{c_1^2}{\chi_h}(Z_i)$ approaches $\frac{4d^2i^2}{d^2i^2} = 4$.

In the case d even the above arguments do not work as $\tau(Z_i) = 0 \bmod 16$. We need a different construction. The idea is, though, the same. Let $\pi : N(d; a, b, m, m) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be a bi-cyclic cover of type $(d, 3)$ branched along (D, C) with $\mathcal{O}(D) = (da, db)$, $\mathcal{O}(C) = (3dm, 3dm)$ and such that D intersects C transversally. To simplify the notation we use $N = N(d; a, b, m, m)$ whenever we want to prove a general statement about the whole class of manifolds. Lemma 3.5 tells us that the canonical line bundle is $K_N = \pi^*(\mathcal{O}((d-1)a+2dm-2, (d-1)b+2dm-2))$, hence N is a surface of general type, with ample canonical line bundle if $(d-1)a + 2m - 2 > 0$, $(d-1)b + 2m - 2 > 0$. Using Lemma 3.6, we can also conclude that N is simply connected.

We want N to be non-spin. We show that this is true if d even and b odd, by finding a class $[A] \in H_2(N, \mathbb{Z})$ such that $[A] \cdot w_2(N) \neq 0 \bmod 2$. We construct N in two steps:

$$N \xrightarrow{\pi_2} X \xrightarrow{\pi_1} \mathbb{CP}^1 \times \mathbb{CP}^1$$

where first we consider a 3-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along C, and then a d-cyclic cover branched along $\pi_1^{-1}(D)$, the proper transform of D.

We construct a 1-parameter family of deformations of N in the following way. Let $D' = A \cup B \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ such that $A = \{pt\} \times \mathbb{CP}^1$, B a smooth curve of bi-degree $(da-1, db)$, and such that A, B, C, D intersect transversally at all points. Let N_0 be the $d-$cover of X branched along $\pi_1^{-1}(D')$. This is singular surface, with $A_{d-1}-$type singularities. To resolve these singularities we introduce strings of exceptional divisors, which we

denote by E_j. We denote the minimal resolution of N_0 by N'. Proposition 3.3 tells us that N' is a complex surface diffeomorphic to N.

We have the following diagram:

$$N' \xrightarrow{\pi_0} N_0 \xrightarrow{\pi_2'} X \xrightarrow{\pi_1} \mathbb{CP}^1 \times \mathbb{CP}^1$$

Let $A'' \subset N'$ be the proper transform of $A' = (\pi_1 \circ \pi_2')^{-1}(A) \subset N_0$. Lemma 3.1 tells us that:

$$\begin{aligned}
\mathcal{O}(A'') &= \frac{1}{d}(\pi_1 \circ \pi_2' \circ \pi_0)^*(\mathcal{O}(1,0)) + \Sigma a_i E_i,\ a_i \in \mathbb{Q},\ a_i \le 0 \\
K_{N'} &= (\pi_1 \circ \pi_2' \circ \pi_0)^*(\mathcal{O}((d-1)a + 2dm - 2, (d-1)b + 2dm - 2)) \\
K_{N'} \cdot A'' &= c_1\Big((\pi_1 \circ \pi_2' \circ \pi_0)^*(\mathcal{O}((d-1)a + 2dm - 2, (d-1)b + 2dm - 2))\Big) \cup \\
&\quad c_1\big(\frac{1}{d}(\pi_1 \circ \pi_2' \circ \pi_0)^*(\mathcal{O}(1,0)) + \Sigma a_i E_i\big) \\
&= \frac{1}{d} 3d c_1\big(\mathcal{O}((d-1)a + 2dm - 2, (d-1)b + 2dm - 2))\big) \cup c_1(\mathcal{O}(1,0)) \\
&= 3((d-1)b + 2dm - 2) \\
&\equiv 1 \bmod 2 \ \text{ if } b \text{ odd},\ d \text{ even}
\end{aligned}$$

Hence N' is non-spin, and so is N.

Let $d = 2^k d',\ d' \in \mathbb{N}$ odd be the decomposition of d.

We want to consider the manifolds:

$$N_i = N(d; 2d', d', i, i) \subset \mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(2d', d') \oplus \mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(di, di) \text{ if } d' \neq 1,$$

$$\text{and} \quad N_i = N(d; 6, 3, i, i) \text{ if } \quad d = 2^k.$$

On these manifolds, we can define a weighted $\mathbb{Z}/d\mathbb{Z}$ action as in Lemma 3.10, where we extend the action trivially on the second factor. For this action to be well-defined we need φ_D and φ_C to be invariant under the induced action on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Such curves always exist. Moreover, we can choose D, C such that $\mathbb{Z}/d\mathbb{Z}$ acts freely on them. The conditions in the Lemma 3.10 are automatically satisfied by our choice of degrees.

Let $Z_i = N_i/(\mathbb{Z}/d\mathbb{Z})$. Z_i is complex surface of general type, with ample canonical line bundle, finite fundamental group $\pi_1(Z_i) = \mathbb{Z}/d\mathbb{Z}$ and of w_2-type (I). Its numerical invariants can be computed using Lemma 3.5 to be:
$c_1^2(Z_i) = 6\big(2(d-1)d' + 2i - 2\big)\big((d-1)d' + 2i - 2\big)$;
$c_2(Z_i) = 3[4 - 2(d-1)(3d' - 2d'^2 d) - 4(2i - 3i^2) + 3(d-1)d'i]$;
$\tau(Z_i) = \frac{1}{3}(c_1^2 - 2c_2)(Z_i) \equiv -6(d-1)d'i \bmod 4$.

For the special case $d = 2^k$ the numerical invariants are computed by the same formulas, for $d' = 3$.

¿From the last relation we see that if i is odd, then $\tau(Z_i) \neq 0 \bmod 8$ hence the intersection form is odd.

We take Z_i to be the subsequence indexed by odd coefficients.

As i increases, $\frac{c_1^2}{\chi_h}(Z_i) = \frac{12c_1^2}{c_1^2+c_2}(Z_i)$ approaches $\frac{12\cdot 6\cdot 4i^2}{6\cdot 4i^2+3\cdot 12i^2} = \frac{24}{5} = 4.8$

Hence considering both cases, d odd or even, there is a constant $n_0 > 0$ such that for any $i \geq n_0$ we have $c_1^2(Z_i) \leq 5\chi_h(Z_i)$. We will re-index our sequence starting from Z_{n_0}.

Moreover, the universal covers of Z_i are almost completely decomposable as they are bi-cyclic covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ (Theorem 3.7). □

4 Proofs of Theorems

As stated in introduction, in Theorem 1.2, for any finite cyclic fundamental group we construct infinitely many classes of manifolds such that each class supports a differential structure that admits an Einstein metric and infinitely many other differential structures which do not admit any Einstein metrics. Moreover some the properties of their universal covers are given. We will give here the proof of the Theorem 1.2:

Proof of Theorem 1.2. The manifolds Z_i are the ones given by Proposition 3.12. They are complex surfaces of general type with ample canonical line bundle. Hence, Aubin-Yau's Theorem on Calabi Conjecture,[Yau77], tells us that these manifolds admit Kähler-Einstein metrics. Moreover, they have odd intersection form and w_2-type (I).

By Theorem 2.3 there exist a constant $n_1 > 0$ such that for any lattice point (x, y) in the first quadrant verifying $x > n_1,\ y \leq 8.5x$ there exists a infinite family of homeomorphic, non-diffeomorphic simply connected minimal symplectic manifolds M'_j such that $y = c_1^2(M'_j),\ x = \chi_h(M'_j) = \frac{(c_1^2+c_2)(M'_j)}{12}$. Eventually after truncating and relabeling the sequence Z_i, we can construct $M'_{i,j},\ i, j \in \mathbb{N}$, a family of simply connected symplectic manifolds satisfying:

1. for fixed i, $M'_{i,j}$ are homeomorphic, but no two are diffeomorphic;
2. $\chi_h(M'_{i,j}) = \chi_h(Z_i)$ for any $j \in \mathbb{N}$;
3. $c_1^2(M'_{i,j}) \geq 8\chi_h(M'_{i,j})$;
4. $c_1^2(Z_i) \leq 5\chi_h(Z_i)$.

Let S_d be the rational homology sphere with fundamental group $\pi_1(S_d) = \mathbb{Z}/d\mathbb{Z}$ and universal cover $\#(d-1)S^2 \times S^2$, as constructed in [Ue96].

The manifolds $M_{i,j}$ are constructed as:

$$M_{i,j} = M'_{i,j}\#S_d\#k\overline{\mathbb{CP}^2}, \quad \text{where } k = c_1^2(M'_{i,j}) - c_1^2(Z_i).$$

We remark that by Theorem 1 [KMT95] the manifolds $M'_{i,j}\#S_d$ are not symplectic, but they have non-trivial Seiberg-Witten invariant. We abuse our notation, and we use $\chi_h(M_{i,j})$ instead of $\frac{\chi+\tau}{4}(M_{i,j})$ and $c_1^2(M_{i,j}) = (2\chi + 3\tau)(M_{i,j})$.

For fixed i the manifolds Z_i and $M_{i,j}$ are all of w_2-type (I), with odd intersection form, fundamental group $\pi_1 = \mathbb{Z}/d\mathbb{Z}$ and have the same numerical invariants: Todd-genus and Euler characteristic. Hence by Theorem 2.1, these manifolds are homeomorphic.

In the construction theorem for $M'_{i,j}$ the differential structures were distinguished by different Seiberg-Witten basic classes. After taking the connected sum with $\overline{\mathbb{CP}^2}$'s and S_d, and using Theorem 1 [KMT95] we see that the Seiberg-Witten basic classes remain different. Hence the manifolds $M_{i,j}$ are not diffeomorphic to one another.

An estimate of the number k of copies of $\overline{\mathbb{CP}^2}$ is given by:
$k = c_1^2(M_{i,j}) - c_1^2(Z_i) \geq 8\chi_h(Z_i) - 5\chi_h(Z_i) = 3\chi_h(Z_i) = 3\chi_h(M_{i,j})$ We also know that the manifolds are under the Bogomolov-Miyaoka-Yau line, which implies: $\chi_h(M_{i,j}) \geq \frac{1}{9}c_1^2(M_{i,j})$.
Hence $k \geq \frac{1}{3}c_1^2(M_{i,j}) = \frac{1}{3}(2\chi + 3\tau)(M_{i,j})$.

Then Theorem 1.1 implies that $M_{i,j}$ does not admit any Einstein metric. As a consequence we also get that Z_i and $M_{i,j}$ are not diffeomorphic.

For the results from the second part of the theorem we have to look at the universal covers $\widetilde{Z}_i$, and $\widetilde{M_{i,j}}$ respectively. From our construction, the universal cover of Z_i is a simply connected minimal complex surface of general type. It can not be diffeomorphic to connected sums of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}^2}$'s as it has non-trivial Seiberg-Witten invariants, but Theorem 3.7 tells us that after connected sum with one copy of $\mathbb{CP}^2$ it decomposes completely.

The universal cover of S_d is diffeomorphic to $(d-1)(S^2 \times S^2)$, hence the manifold $\widetilde{M_{i,j}} \cong dM'_{i,j}\#dk\overline{\mathbb{CP}^2}\#(d-1)(S^2 \times S^2)$. But $(S^2 \times S^2)\#\overline{\mathbb{CP}^2}$ is the complex surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown-up at one point, which can also be presented as $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$. So:

$$\widetilde{M_{i,j}} \cong dM'_{i,j}\#dk\overline{\mathbb{CP}^2}\#(d-1)S^2 \times S^2 \cong dM'_{i,j}\#(d-1)\mathbb{CP}^2\#(dk+d-1)\overline{\mathbb{CP}^2}.$$

The manifolds $M'_{i,j}$ are almost completely decomposable, hence $\widetilde{M_{i,j}}$ is diffeomorphic to the connected sums of a number of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}^2}$'s. $\square$

Proof of Proposition 1.3. Let N be the double cover of $\mathbb{CP}^2$ branched along a smooth divisor D, such that $\mathcal{O}(D) = \mathcal{O}_{\mathbb{CP}^2}(8)$. Then N is a simply connected, almost completely decomposable surface of general type and by Lemma 3.2 its numerical invariants are:

$$c_2(N) = 46,\ c_1^2(N) = 2,\ \tau(N) = -30.$$

Hence $b_2^+ = 7$ and $b_2^- = 37$. By Theorem 1.1 the manifold $M = N\#\overline{\mathbb{CP}^2}\#S_2$ does not admit any Einstein metric.

Let $\widetilde{M}$ be the universal cover of M. Then we have the following diffeomorphisms:

$$\widetilde{M} \cong 2N\#2\overline{\mathbb{CP}^2}\#(S^2 \times S^2) \cong 2N\#\mathbb{CP}^2\#3\overline{\mathbb{CP}^2} \cong 15\mathbb{CP}^2\#77\overline{\mathbb{CP}^2}.$$

As M does not admit any Einstein metrics this implies that $\widetilde{M}$ does not admit any Einstein metrics invariant under the covering involution σ. $\square$

Proof of Proposition 1.4. Let $M' = N\#\overline{\mathbb{CP}^2}\#S_3$, where the manifold N is the same as in the proof of the previous proposition and S_3 is a rational sphere.

Then the universal cover $\widetilde{M'}$ is diffeomorphic to

$$\widetilde{M'} \cong 3N\#3\overline{\mathbb{CP}^2}\#2(S^2 \times S^2) \cong 3N\#2\mathbb{CP}^2\#5\overline{\mathbb{CP}^2} \cong 23\mathbb{CP}^2\#116\overline{\mathbb{CP}^2}.$$

$\square$

We can combine the idea of the above propositions with the geography of almost completely decomosable symplectic manifolds 2.3 to proof a general result:

Proof of Theorem 1.5. Let $\epsilon > 0$ be a small positive number.

Theorem 2.3 tells us that for any $\epsilon' > 0$ there exists an $c(\epsilon')$ such that to any integer point (x, y) in the latice:

$$0 < y \leq (9 - \epsilon')x - c(\epsilon')$$

we can associate infinitely many homeomorphic, non-diffeomorphic almost completely decomposable minimal symplectic manifolds which have topological invariants $(\chi_h, c_1^2) = (x, y)$. Let $\epsilon' = \frac{3}{2}\epsilon$ then there exists an $c(\epsilon')$ with the above properties. Let $N(\epsilon) = \frac{2d}{3}(c(\epsilon') + 1)$.

Let n, m integer points such that the conditions $(1-3)$ in Theorem 1.5 are satisfied. Condition 3 states the following:

$$n \quad < \quad (6 - \epsilon)m - N(\epsilon)$$

Or equivalently:

$$\begin{aligned}
\frac{n}{d} &< (6-\epsilon)\frac{m}{d} - \frac{N(\epsilon)}{d} \\
\frac{3n}{2d} &< (9-\frac{3}{2}\epsilon)\frac{m}{d} - \frac{3N(\epsilon)}{2d} \\
\frac{3n}{2d} &< (9-\epsilon')\frac{m}{d} - c(\epsilon') - 1 \\
\frac{3n}{2d} + 1 &< (9-\epsilon')\frac{m}{d} - c(\epsilon')
\end{aligned}$$

Then $[\frac{3n}{2d}] + 1$ and $\frac{m}{d}$ satisfy the conditions of Theorem 2.3, hence there are infinitely many symplectic manifolds M_i such that $c_1^2(M_i) = [\frac{3n}{2d}] + 1$ and $\frac{\chi+\tau}{4}(M_i) = \frac{m}{d}$. Moreover M_i are homeomorphic, non-diffeomorphic almost completely decomposable manifolds. The differential structures are distinguished by the difference in the Seiberg-Witten basic classes.

Let:

$$X_i = M_i \# S_d \# k\overline{\mathbb{CP}^2}, \quad k = [\frac{3n}{2d}] + 1 - \frac{n}{d} > \frac{1}{3}c_1^2(M_i)$$

Then $\frac{\chi+\tau}{4}(X_i) = \frac{m}{d}$ and $(2\chi+3\tau)(X_i) = \frac{n}{d}$. The manifolds X_i remain homeomorphic to each other, and using the formula for the Seiberg-Witten basic classes for the connected sum we can immediately see that any two manifolds are not diffeomorphic. Their universal cover $\widetilde{X_i}$ is diffeomorphic to $dX_i \# dk\overline{\mathbb{CP}^2} \# (d-1)(S^2 \times S^2)$ and as X_i are almost completely decomposable, this implies that all $\widetilde{X_i}$ are diffeomorphic to $X = a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2}$ for suitable a, b such that $\frac{\chi+\tau}{4}(X) = m, (2\chi+3\tau)(X) = n$.

As $M_i \# S_d$ has non-trivial Seiberg-Witten invariants, we can use Theorem 1.1 to conclude that the manifolds X_i don't admit an Einstein metric. Hence their universal cover X does not admit any Einstein metric invariant under any of the $\mathbb{Z}/d\mathbb{Z}-$actions. □

Proof of Theorem 1.6. The proof is again based on the obstruction given by Theorem 1.1. For constructing our examples we employ Gompf's techniques, [Gom95], and use symplectic connected sum along symplectic submanifolds of self-intersection 0. We need some standard blocks of symplectic manifolds.

The first block, we denote by X_G, and it is the spin symplectic 4-manifolds constructed by Gompf (see [Gom95] Theorem 6.2.). X_G has fundamental group G, $c_1^2(X_G) = 0$ and contains, as a symplectic submanifold, a $2-$torus of self-intersection 0.

The second block, denoted by $E(n)$, is the family of simply connected, proper elliptic complex surfaces, with no multiple fibers and Euler characteristic $\chi(E(n)) = 12n$. A generic fiber is a symplectic torus of self-intersection 0, and it can easily be checked,[GoSt99], that its complement is simply connected.

The third block is the one used by Gompf [Gom95] in the proof of Theorem 6.1. Let F_1, F_2 be two Riemann surfaces of genera $k+1$ ($k \geq 1$) and 2, respectively. Let $C_i,\ i = 1, \ldots, 2k+2$ be homologically nontrivial embedded circles in F_1 generating $H_1(F_1, \mathbb{Z})$ and the circles $C'_i \subset F_2,\ i = 1, \ldots, 4$ be the generators of $H_1(F_2, \mathbb{Z})$, such that $C_{2i} \cap C_{2j} = C_{2i-1} \cap C_{2j-1} = \emptyset$ if $i \neq j$ and $C_{2i} \cdot C_{2j-1} = \delta_i^j,\ i, j = \overline{1, k+1}$. C'_i are also satisfying the above conditions. Let T_i be the collection of tori given by $C_1 \times C'_1, C_2 \times C'_3, C_3 \times C'_2, C_4 \times C'_4, C_i \times C'_1,\ i = 5, \ldots, 2k+2$. We can perturb this collection to a new collection of *disjoint* tori $\{T'_i\}$, where T'_i is homologous to T_i. As $T_i \subset F_1 \times F_2$ is a Lagrangian torus we may choose $T'_i \subset F_1 \times F_2$ to be Lagrangian, too. Moreover these tori are also homologically non-trivial, hence we can perturb the product symplectic form on $F_1 \times F_2$, see [Gom95], such that these tori become symplectic submanifolds. Let X_k be the manifold obtained by performing symplectic connected sum of $F_1 \times F_2$ and $2k+2$ copies of $E(2)$ along the family $\{T'_i\}_{i=\overline{1,2k+2}}$ and generic fibers of $E(2)$. Then the manifold X_k is a spin, symplectic 4-manifold, and by Seifert-Van Kampen Theorem it is also simply connected. The numerical invariants of X_k are $\chi(X_k) = 52k + 48,\ \tau(X_k) = -32(k+1)$.

The fourth block has a linking role, and it is $E(4)$ with a special symplectic structure. This manifold has an important feature [Gom95, proof of Theorem 6.2]: it contains a torus and a genus 2 surface as disjoint symplectic submanifolds. We denote them by T and F respectively. Both T and F have self-intersection zero and their complement $E(4) \setminus (F \cup T)$ is simply connected.

We are now ready to construct our symplectic manifolds. Let:

$$M_i = X_G \ \#_{T^2}\ E(4)\ \#_{\Sigma_2}\ X_i\ \#_{\Sigma_2}\ E(4)\ \#_{T^2}\ E(2) \# p\overline{\mathbb{CP}^2},$$

where $\#_{T^2}$'s are the symplectic sums along tori of self-intersection zero and $\#_{\Sigma_2}$'s are fiber sums along Riemann surfaces of genus 2, represented by $F \subset E(4)$ and one generic $\{pt\} \times F_2 \subset X_i$. The last operation is simply a connected sum and p is a constant satisfying:

$$(c_1^2(X_i) + 16)\ >\ p\ > \frac{1}{3}(c_1^2(X_i) + 16)\ > 0.$$

The fundamental group of M_i can be easily computed by Seifert-Van Kampen Theorem to be G.

To obtain different differential structures on M_i, we take logarithmic transformations of different multiplicities along a generic fiber of $E(2)$. Then by the gluing formula for the Seiberg-Witten invariants [Par02] (Cor 15,20) the manifolds have different Seiberg-Witten invariants. Hence we have constructed infinitely many non-diffeomorphic manifolds. We denote them by $M_{i,j}$.

For fixed i, these manifolds are all homeomorphic. To show this, we can first do the logarithmic transformations on $E(2)$, this yields homeomorphic manifolds. By taking the fiber sum along a generic fiber with the remaining terms we obtain homeomorphic manifolds.

Theorem 1.1 implies that no $M_{i,j}$ admits an Einstein metric. □

All the manifolds constructed up to now are non-spin. If we want to analyze the spin case that we need to use different obstructions. Such obstruction were found by LeBrun and Ishida. They are, though, for a different class of manifolds, which is constructed as connected sums.

Taubes showed that the Seiberg-Witten invariant vanishes for manifolds which decompose as connected sums of manifolds with positive b_2^+. A refinement of the Seiberg-Witten invariant defined independently by Bauer and Furuta is needed. Using similar estimates and this new invariant Ishida and LeBrun were able to prove:

Theorem 4.1. [IsLe02] *Let X_j, $j = 1, \dots, 4$ be smooth, compact almost-complex 4-manifolds for which the mod-2 Seiberg-Witten invariant is non-zero, and suppose that*

$$\begin{aligned} b_1(X_j) &= 0, & (2)\\ b_2^+(X_j) &\equiv 3 \bmod 4, & (3)\\ \sum_{j=1}^4 b_2^+(X_j) &\equiv 4 \bmod 8. & (4) \end{aligned}$$

Let N be any oriented 4-manifold with $b_2^+ = 0$. Then, for any $m = 2, 3$ or 4, the smooth 4-manifold $M = \#_{j=1}^m X_j \# N$ does not admit Einstein metrics if

$$4m - (2\chi + 3\tau)(N) \geq \frac{1}{3} \sum_{j=1}^m c_1^2(X_j).$$

Proof of Theorem 1.7. The manifolds M_i are constructed as connected sums and fiber sums of different blocks.

For the first block we start with X_G, the spin symplectic 4-manifolds constructed by Gompf. Taking the symplectic connected sum with the elliptic surface $E(2n)$ along arbitrary fibers ($\{pt\} \times T^2 \cong F_0 \subset E(2n)$) we obtain a new symplectic, spin manifold which we denote by $N_G(n)$. The elliptic surface $E(2n)$ that we consider has no multiple fibers and its numerical invariants are $c_2(E(2n)) = 24n,\ \tau(E(2n)) = -16n,\ c_1^2(E(2n)) = 0$. Moreover the complement of the generic fiber F_0 is simply connected. Hence the manifold $N_G(n)$ satisfies the following: $\pi_1(N_G(n)) = G,\ c_1^2(N_G(n)) = 0,\ c_2(N_G(n)) = 24k+24n$. Since G is finite, $b_1(N_G(n)) = 0$, and $b_2^+(N_G(n)) = 4(k+n)-1 \equiv 3 \mod 4$.

The second block is obtained from $E(2)$ after performing a logarithmic transformation of order $2n+1$ on one non-singular elliptic fiber. We denote the new manifolds by Y_n. All Y_n are simply connected spin manifolds with $b_2^+ = 3$ and $b_2^- = 19$, hence they are all homeomorphic. Moreover, Y_n are Kähler manifolds and $c_1(Y_n) = 2nf$, where f is the multiple fiber introduced by the logarithmic transformation (see [BPV84]). Hence $\pm 2nf$ is a basic class, and its Seiberg-Witten invariant is ± 1.

The third block is a "small" spin manifold, and we used it in the proof of the previous theorem, X_2. Its invariants are $c_2(X_2) = 152,\ \tau(X_2) = -96,\ c_1^2(X_2) = 16,$ and $b_2^+ = 27\ (\equiv 3 \mod 4)$.

We may now define our manifolds:

$$M_{i,j} = X_2 \# N_G(i) \# Y_j.$$

For fixed i, the manifolds $M_{i,j}$ are all homeomorphic as we take connected sums of homeomorphic manifolds. We denote this homeomorphism type by M_i.

If we consider the basic classes of the Bauer-Furuta invariant, then both $a = c_1(X_1) + c_1(N_G(i)) + c_1(Y_j)$ and $b = c_1(X_1) + c_1(N_G(i)) - c_1(Y_j)$ are basic classes. Then $4j \mid (a - b)$. But any manifold has a finite number of basic classes which are a diffeomorphism invariant. As we let j take infinitely many values, this will imply that $M_{i,j}$ represent infinitely many types of diffeomorphism classes. For a more detailed explanation we reffer the reader to [IsLe03].

By Theorem 4.1 these manifolds do not support any Einstein metrics, but they satisfy Hitchin-Thorpe Inequality. $\square$

Proof of Theorem 1.8. We begin by constructing a simply connected, spin manifolds with small topological invariants and $b_2^+ \equiv 3 \mod 4$. One such manifold is given by a smooth hypersurface of tridegree $(4,4,2)$ in $\mathbb{CP}^1 \times$

$\mathbb{CP}^1 \times \mathbb{CP}^1$, which we denote by X. Its numerical invariants can be easily computed to be $c_1^2(X) = 16$, $c_2(X) = 104$, $b_2^+(X) = 19$ and it is simply connected. Then by Freedman's Theorem, X is homeomorphic to $4K3\#7(S^2 \times S^2)$. A result of Wall [Wa64] tells us that there exists an integer n_0 such that $X\#n_0(S^2 \times S^2)$ becomes diffeomorphic to $4K3\#7(S^2 \times S^2)\#n_0(S^2 \times S^2)$.

Assuming the notation from the previous theorem, let:

$$M_{1,n}^j = X\#Y_j\#E(2n)\#S_d$$

$$M_{2,n}^j = X\#E(2)\#Y_j\#E(2(2n-1))\#S_d$$

By Theoren 4.1 the above manifolds do not admit an Einstein metric.

The manifolds $\{M_{1,n}^j \mid j \in \mathbb{N}\}$ are all homeomorphic, but they represent infinitely many differential structures. Moreover, if we consider their universal cover, $\widetilde{M_{1,n}^j}$ is diffeomorphic to $dX\#dY_j\#dE(2n)\#(d-1)(S^2 \times S^2)$. But Mandelbaum [Ma80] proved that both Y_j and $E(2n)$ completely decompose as connected sums of $K3'$s and $S^2 \times S^2$'s after taking the connected sum with one copy of $S^2 \times S^2$. Hence, for $d > n_0$, the manifold $\widetilde{M_{1,n}^j}$ is diffeomorphic to $d(4K3\#7(S^2 \times S^2))\#d(K3)\#d(nK3\#(n-1)(S^2 \times S^2))\#(d-1)(S^2 \times S^2)$ i.e. to $d(n+5)K3\#(d(n+7)-1)(S^2 \times S^2) = M_{1,n}$.

Remark that the diffeomorphism type of the universal cover does not depend on j. Hence on $M_{1,n}$ we have constructed infinitely many non-equivalent, free actions of $\mathbb{Z}/d\mathbb{Z}$, such that there is no Einstein metric which is invariant under any of the group actions. But all $M_{1,n}$ satisfy the Hitchin-Thorpe Inequality.

Redoing the same arguments for the second example $M_{2,n}^j$, gives us the results for the second family of manifolds. □

References

[BPV84] W. Barth, C. Peters, A. V. de Ven, 'Compact Complex Surfaces', Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4, Springer Verlag, Berlin, 1984.

[BaFu04] S. Bauer , M. Furuta, 'A stable cohomotopy refinement of Seiberg-Witten invariants: I', Invent. Math. 155 (2004), no.1, 1–19.

[Bes87] A. Besse, 'Einstein Manifolds', Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer Verlag, Berlin, 1987.

[BrKo05] V. Braungardt, D. Kotschick, 'Einstein metrics and the number of smooth structures on a four-manifold', Topology 44 (2005), no. 3, 641–659.

[Cat84] F. Catanese, 'On the moduli spaces of surfaces of general type', J. Differential Geom. 19 (1984), no. 2, 483–515.

[Cat92] F. Catanese, '(Some) old and new results on algebraic surfaces', First European Congress of Mathematics, Vol. I (Paris, 1992), Progr. Math., 119, Birkhuser, Basel, 1994, 445–490.

[Par02] D.B. Park, 'A gluing formula for the Seiberg-Witten invariant along T^3', Michigan Math. J. 50 (2002), no. 3, 593–611

[Fre82] M. Freedman, 'On the topology of 4-manifolds', J. Differential Geom.17 (1982), no.3, 357–453.

[FrMo94] R. Friedman, J. Morgan, 'Smooth four-manifolds and complex surfaces', Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 27, Springer-Verlag, Berlin, 1994.

[Gom95] R. E. Gompf, 'A new construction of symplectic manifolds', Ann. of Math. (2) 142 (1995), no. 3, 527–595.

[GoSt99] R. E. Gompf, A. I. Stipsicz, '4-manifolds and Kirby calculus', American Mathematical Society, Providence, RI, 1999.

[HaKr93] I. Hambleton, M. Kreck, 'Cancellation,elliptic surfaces and the topology of certain four-manifolds', J. Reine Angew. Math., 444 (1993), 79–100.

[HKK86] J. Harer, A. Kas, R. Kirby, 'Handlebody decompositions of complex surfaces', Mem. Amer. Math. Soc. 62 (1986), no. 350.

[Har77] R. Hartshorne , 'Algebraic geometr', Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[IsLe02] M. Ishida and C. LeBrun, 'Spin manifolds, Einstein metrics, and differential topology', Math. Res. Lett., 9 (2002), no.2-3, 229-240.

[IsLe03] M. Ishida and C. LeBrun, 'Curvature, connected sums, and Seiberg-Witten theory', Comm. Anal. Geom. 11 (2003), no. 5, 809–836.

[Kot98] D. Kotschick, 'Einstein metrics and smooth structures', Geom. Topol., 2 (1998), 1–10.

[KMT95] D. Kotschick, J. W. Morgan, C. H. Taubes, 'Four-manifolds without symplectic structures but with nontrivial Seiberg-Witten invariants', Math. Res. Lett. 2 (1995), no. 2, 119–124.

[LeB96] C. LeBrun, 'Four-manifolds without Einstein metrics', Math. Res.Lett. 3 (1996), no. 2, 133–147.

[LeB01] ——, 'Ricci curvature, minimal volumes, and Seiberg-Witten theory', Invent. Math. 145 (2001), no. 2, 279–316.

[LeB02] ——, 'Einstein metrics, four-manifolds, and differential topology', Surveys in differential geometry, Vol. VIII , International Press, Boston, MA, 2002, 235–255.

[MaMo80] R. Mandelbaum, B. Moishezon, 'On the topology of simply connected algebraic surfaces', Trans. Amer. Math. Soc. 260 (1980), no. 1, 195–222.

[Ma80] R.Mandelbaum, 'Four-dimensional topology: an introduction', Bull.Amer.Math.Society, 2 (1980), 1–159.

[Tau94] C. H. Taubes, 'The Seiberg-Witten invariants and symplectic forms', Math. Res. Lett. 1 (1994), no. 6, 809–822.

[Ue96] M. Ue, 'A remark on the exotic free actions in dimension 4', J. Math. Soc. Japan 48 (1996), no. 2, 333–350.

[Wa64] C. T. C. Wall 'On simply-connected 4-manifolds', J. London Math. Soc. 39 1964 141–149

[Wit94] E. Witten 'Monopoles and four-manifolds', Math. Res. Lett. 1 (1994), no. 6, 809–822.

[Yau77] S.-T. Yau, 'Calabi's conjecture and some new results in algebraic geometry'. Proc. Nat. Acad. USA 74 (1997), 1789–1799.